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LETTER TO THE EDITOR

On certain local observables generated by the momenta

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Abstract. We study the quantisation of certain local quantum observables defined in terms of a momentum and a region of a Riemannian configuration space. Geometric criteria are given for the quantisability of these local observables, conditions are developed under which a reconstruction of the corresponding global quantum momentum may be attempted, and explicit formulae describing such a reconstruction are given.

In a recent paper (Wan and McFarlane 1980) we established a link between the geometric quantisability of a momentum $\boldsymbol{P}$ defined over a Riemannian configuration manifold $\boldsymbol{M}$ and certain global bounds in the accuracy of measurements of $\boldsymbol{P}$ conducted upon localised states within a finite region of space; momenta possessing such bounds were then said to be globally measurable. The key features of this analysis were the localisation of measurement within finite regions of the configuration space, and the combination of locally-derived measures to deduce global properties of the momentum under test. Localisation was in the case of quantum measurement achieved by a suitable *a priori* restriction of wavefunctions to those whose support was contained in the local region under consideration.

There is, however, a second means whereby localisation can be introduced, and this without the restriction of admissible states: we may seek to define local observables derived from the global momenta such as need be measured only within a given volume of space. Intuitively we may expect such an observable to satisfy two basic *desiderata*:

(i) It is, outside a certain region, identically zero, so that no apparatus external to that locality is required to determine its value, and

(ii) it agrees within that certain region with a global observable from which it is derived. Note that the second point should not be taken too literally since a well-defined global quantum momentum need not generally exist.

We shall be concerned in this Letter with the properties of such local observables and their relationship to the momenta from which they are derived, and shall find that whereas local momenta do not in general effect a means of decomposing their parent observables into simpler and localised objects suitable for measurement, whenever they exist they nevertheless define a canonical decomposition of the global momentum, and may, significantly, be well defined even for those classical observables for which no quantum counterpart exists.

Now let $\boldsymbol{A}$ be an open subset of $\boldsymbol{M}$ which may be visualised as a small region of $\boldsymbol{M}$ within which we wish to conduct measurements, and let π denote the projector onto $\boldsymbol{A}$

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whose action on an element ψ of the natural Hilbert space $\boldsymbol{L}^2(\boldsymbol{M})$ is given as

$$\pi\psi(x)=\begin{cases}\psi(x), & x\in\boldsymbol{A},\\ 0, & x\notin\boldsymbol{A}.\end{cases}$$

We may then introduce, at least formally, a local quantum observable defined over $\boldsymbol{A}$ and associated with the momentum $\boldsymbol{P}$ by the formula

$$\boldsymbol{Q}_0^{\boldsymbol{A}}(\boldsymbol{P})=\pi\boldsymbol{Q}_0(\boldsymbol{P})\pi,$$

in which $\boldsymbol{Q}_0(\boldsymbol{P})$ denotes the usual formal quantum analogue of $\boldsymbol{P}$ (Wan and McFarlane 1980),

$$\boldsymbol{Q}_0(\boldsymbol{P})=-\mathrm{i}\hbar(\boldsymbol{X}+\tfrac{1}{2}\operatorname{div}\boldsymbol{X}),$$

defined on the domain $\boldsymbol{C}_0^\infty(\boldsymbol{M})$, in which $\boldsymbol{X}$ denotes the vector field associated with $\boldsymbol{P}$, and $\boldsymbol{C}_0^\infty(\boldsymbol{M})$ the set of infinitely differentiable functions of compact support. Clearly $\boldsymbol{Q}_0^{\boldsymbol{A}}(\boldsymbol{P})$ is a symmetric operator, and hence obeys our first *desideratum* trivially. The second, however, gives rise to difficulties since in general the domain $\boldsymbol{DQ}(\boldsymbol{P})$ of any self-adjoint extension of $\boldsymbol{Q}_0(\boldsymbol{P})$ may not contain $\boldsymbol{DQ}^{\boldsymbol{A}}(\boldsymbol{P})$, the domain of any self-adjoint extension of $\boldsymbol{Q}_0^{\boldsymbol{A}}(\boldsymbol{P})$, so that a state ψ possessing the global quantum attribute $\boldsymbol{Q}(\boldsymbol{P})$ may possess no local attribute $\boldsymbol{Q}^{\boldsymbol{A}}(\boldsymbol{P})$ at all! Furthermore $\boldsymbol{Q}_0^{\boldsymbol{A}}(\boldsymbol{P})$ may possess no self-adjoint extension, and examples may easily be constructed to demonstrate this.

Resuming our discussion, let us consider, for the case of the complete momenta (that is momenta associated with complete vector fields) the source of the difficulties with our second *desideratum*. For such momenta there exists a unique self-adjoint extension

$$\boldsymbol{Q}(\boldsymbol{P})=-\mathrm{i}\hbar(\boldsymbol{D}_{\boldsymbol{X}}+\tfrac{1}{2}\operatorname{div}\boldsymbol{X}),$$

having the domain of definition

$$\boldsymbol{DQ}(\boldsymbol{P})=\{\psi\in\boldsymbol{L}^2(\boldsymbol{M})|\psi\in\boldsymbol{C}^1(\boldsymbol{X},\boldsymbol{M}),\ \boldsymbol{Q}(\boldsymbol{P})\psi\in\boldsymbol{L}^2(\boldsymbol{M})\},$$

in which $\boldsymbol{D}_{\boldsymbol{X}}$ denotes the Lie derivative with respect to the field $\boldsymbol{X}$, and $\boldsymbol{C}^1(\boldsymbol{X},\boldsymbol{M})$ the set of functions ψ for which $\boldsymbol{D}_{\boldsymbol{X}}\psi$ exists (Wan and McFarlane 1980). Observe that $\boldsymbol{Q}_0^{\boldsymbol{A}}(\boldsymbol{P})$ has the symmetric extension $\pi\boldsymbol{Q}(\boldsymbol{P})\pi$ and that, provided $\pi\psi\in\boldsymbol{C}^1(\boldsymbol{X},\boldsymbol{M})$, $\boldsymbol{Q}(\boldsymbol{P})\psi\in\boldsymbol{L}^2(\boldsymbol{M})\Rightarrow\pi\boldsymbol{Q}(\boldsymbol{P})\pi\psi\in\boldsymbol{L}^2(\boldsymbol{M})$, so that we are led to suspect that the problem with the second *desideratum* stems from the failure of the domain constraint $\pi\psi\in\boldsymbol{C}^1(\boldsymbol{X},\boldsymbol{M})$ even when $\psi\in\boldsymbol{C}^1(\boldsymbol{X},\boldsymbol{M})$. But now $\psi\in\boldsymbol{C}^1(\boldsymbol{X},\boldsymbol{M})$ does not imply that $\pi\psi\in\boldsymbol{C}^1(\boldsymbol{X},\boldsymbol{M})$ without constraint on ψ unless when, as may best be seen by geometrical visualisation, $\boldsymbol{A}$ is invariant under the flow of $\boldsymbol{X}$, so that we anticipate that only those sets $\boldsymbol{A}$ which are invariant under the flow of $\boldsymbol{X}$ will generate satisfactory local quantum observables, a speculation confirmed by the following theorem:

Theorem 1. On the existence of local observables. The operator $\boldsymbol{Q}_0^{\boldsymbol{A}}(\boldsymbol{P})$ is essentially self-adjoint if and only if the flow σ_t of $\boldsymbol{X}$ is complete in $\boldsymbol{A}$, except possibly on a set of measure zero†, when the unique self-adjoint extension $\boldsymbol{Q}^{\boldsymbol{A}}(\boldsymbol{P})$ is given by

$$\boldsymbol{Q}^{\boldsymbol{A}}(\boldsymbol{P})\psi(x)=\begin{cases}-\mathrm{i}\hbar(\boldsymbol{D}_{\boldsymbol{X}}+\tfrac{1}{2}\operatorname{div}\boldsymbol{X})\psi(x), & x\in\boldsymbol{A},\\ 0, & x\notin\boldsymbol{A},\end{cases}$$

† By this we mean that at almost every point of $\boldsymbol{A}$ the maximal integral curve $\{y\in\boldsymbol{M}|\sigma_t(x)=y\}$ has an infinite range of t parametrisation, and moreover $\sigma_t(x)\in\boldsymbol{A}$, $\forall t\in R$. We may also term $\boldsymbol{A}$ invariant under the flow of $\boldsymbol{X}$. Note especially that $\boldsymbol{X}$ need not be complete in $\boldsymbol{M}$.

on the domain

$$DQ^A(P) = \{\psi \in L^2(M) | \psi \in C^1(X, A), Q^A(P)\psi \in L^2(M)\},$$

in which $C^1(X, A)$ denotes the set of functions on M whose Lie derivative with respect to X exists at all points of A.

Proof. We first establish a lemma. Let Ω_0 be a symmetric restriction of a symmetric operator Ω; then Ω is essentially self-adjoint if Ω_0 is essentially self-adjoint. For we have $\Omega_0 \leqslant \Omega_0^{\dagger\dagger} \leqslant \Omega_0^{\dagger}$, $\Omega \leqslant \Omega^{\dagger\dagger} \leqslant \Omega^{\dagger}$ and $\Omega_0 < \Omega$ whence $\Omega_0^{\dagger} \geqslant \Omega^{\dagger}$, $\Omega_0^{\dagger\dagger} \leqslant \Omega^{\dagger\dagger}$ and therefore $\Omega_0^{\dagger} = \Omega_0^{\dagger\dagger}$ requires $\Omega^{\dagger} = \Omega^{\dagger\dagger}$. Next define the auxiliary operator q_0^A on the Hilbert space $L^2(A)$ by $q_0^A = -i\hbar(X + \frac{1}{2}\operatorname{div} X)$ on the domain $C_0^\infty(A)$ of $L^2(A)$. Treating $A \subset M$ as a sub-manifold of M it is immediate that q_0^A as an operator in $L^2(A)$ is essentially self-adjoint if and only if the conditions of the above theorem hold (Abraham and Marsden 1978, Wan and McFarlane 1980). Further define on $L^2(M)$ the operator $\bar{Q}_0^A = q_0^A \oplus \Theta_\perp^A$ where $\Theta_\perp^A$ denotes the zero operator on the orthogonal complement $L_\perp^2(A)$ of $L^2(A)$ regarded as a subspace of $L^2(M)$ so that $L^2(M) = L^2(A) \oplus L_\perp^2(A)$. We then have $(\bar{Q}_0^A)^\dagger = (q_0^A)^\dagger \oplus \Theta_\perp^A$, and hence that $\bar{Q}_0^A$ is essentially self-adjoint if and only if so also is q_0^A. But now $Q_0^A(P)$ is a symmetric extension of $\bar{Q}_0^A$, and is itself symmetric so that by our lemma if $Q_0^A(P)$ is essentially self-adjoint then so also is $\bar{Q}_0^A$ and hence the conditions on σ_t are satisfied. Conversely if the conditions are satisfied then $Q_0^A(P)$ is essentially self-adjoint; for $(Q_0^A(P))^{\dagger\dagger} \geqslant \bar{Q}_0^A$ so that by the lemma $(Q_0^A(P))^{\dagger\dagger\dagger\dagger} = (Q_0^A(P))^{\dagger\dagger\dagger}$ or $(Q_0^A(P))^{\dagger\dagger} = (Q_0^A(P))^{\dagger}$. Finally when the conditions of the theorem are satisfied we have that $Q^A(P) = (\bar{Q}_0^A)^\dagger = (q_0^A)^\dagger \oplus \Theta_\perp^A$ and $(q_0^A)^\dagger = -i\hbar(D_X + \frac{1}{2}\operatorname{div} X)$ acting on functions of $L^2(A)$ for which the Lie derivative exists. This leads directly to the explicit expression for $Q_A(P)$.

Observe that although a local observable $Q^A(P)$ may exist even when no global observable $Q(P)$ is defined, the existence condition for such a local observable, that the set A be invariant under the flow σ_t, is nonetheless stringent, and for some momenta P no such local A may exist. At the other extreme we may have a momentum for which a family of σ_t- invariant sets A_α may be defined such that $\{A_\alpha\}$ is a partition of M, except for a set of measure zero. Clearly in this case $Q(P)$ is well defined, since P is complete almost everywhere, and can be given a decomposition in terms of the corresponding local observables as follows:

Theorem 2. On the canonical decomposition of global observables. Let P admit a family of (almost everywhere) invariant open subsets A_α which partition M except for a set of measure zero; then P is complete almost everywhere, and moreover

$$Q(P) = \sum_\alpha Q^{A_\alpha}(P).$$

Proof. The domain $D(\Sigma_\alpha Q^{A_\alpha}(P))$ contains $\phi \in \bigcap_\alpha C^1(X, A_\alpha)$ which implies that $\phi \in C^1(X, M)$ almost everywhere. Further, since in $L^2(M)$ we do not distinguish functions differing only in a set of measure zero, we have $\phi \in D(\Sigma_\alpha DQ^{A_\alpha}(P)) \Rightarrow \phi \in C^1(X, M)$. Now obviously $\forall \phi \in D(\Sigma_\alpha Q^{A_\alpha}(P))$, we have $\Sigma_\alpha Q^{A_\alpha}(P)\phi = Q(P)\phi$ almost everywhere, and conversely $\forall \phi \in DQ(P)$, so that the result follows.

References

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